Iranian Journal of Educational Research

Print ISSN: 1735 - 563X Online ISSN: 2980 - 874X

Homepage: http://ijer.hormozgan.ac.ir

University of Hormozgan

Educational and Behavioral Research Center

Graph Theory Students' Perceptions of Graph Isomorphism

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Article Info

Article type:

Research Article

Article history:

Received 05 Jul. 2024

Received in revised form 11 Sep. 2024

Accepted 11 Oct. 2024

Published online 01 Mar. 2025

Keywords:

Graph Theory,
Isomorphism,
Conceptual Metaphors,
Variation

ABSTRACT

Objective: Graph isomorphism is a central topic in graph theory, yet students often struggle to grasp its underlying principles. This study aimed to explore students' perceptions of graph isomorphism and identify the conceptual metaphors they employ when reasoning about this concept.

Methods: Using a two-stage qualitative research design, the study combined surveys and task-based interviews to collect data. The variation theory framework (Marton & Booth, 2013) was applied to analyze students' understanding and identify critical and superficial features in their reasoning.

Results: The findings revealed that students frequently focus on superficial features, such as the equality of the number of vertices and edges, while overlooking critical features like bijective attributes. Additionally, the conceptual metaphors used by students were categorized into three main themes: Sameness, Sameness/Mapping, and Formal Definition. These metaphors reflect varying levels of conceptual understanding, with some students relying on intuitive notions of similarity and others employing more formal, mathematical definitions.

Conclusions: This study highlights the challenges students face in understanding graph isomorphism and underscores the importance of addressing both superficial and critical features in teaching. By identifying the conceptual metaphors students use, educators can develop targeted instructional strategies to bridge gaps in understanding and promote deeper conceptual learning.

Cite this article: Jafari, A., Haghverdi, M. & Tehranian, A. (2025). Graph theory students' perceptions of graph isomorphism. *Iranian Journal of Educational Research*, 4 (1), 155-172.

DOI: https://doi.org/10.22034/4.1.155

DOI: https://doi.org/10.22034/4.1.155

Introduction

Researchers have introduced the graph isomorphism as one of the most important topics in graph theory. Gross (2019). Although some studies have been done on the applications and theorems of graph isomorphism, less importance has been given to students' perception of the concept of graph isomorphism. The analysis of students' perception of the concept of graph isomorphism can help to identify the learning routes for students.

Researchers have examined conceptions of group isomorphism more than graph isomorphism .For example, the study of Rapnow (2021) who categorized conceptual metaphors related to the isomorphism of groups.

The graph isomorphism problem (GI) is that of determining whether an isomorphism between two given graphs? GI has long been a favorite target of algorithm designers, so much so that it was already described as a “disease” in 1976 (Read and Corneil,1977). Although presented algorithms are perfectly accurate, their complexity is not so ideal and may become exponential in some special scenarios. In fact, most classical graph isomorphism algorithms have a high computational complexity. Therefore, researchers seek to explore an efficient graph isomorphism algorithm with polynomial time complexity.

In general, researchers make occasional nods to isomorphism importance, difficulty and applications, such as in Somkunwar’s (2017) study identifying that graph isomorphism various applications, such as image processing, protein structure, social networks and chemical structure.

From this set of research, it can be concluded that researchers have given less importance to students' perceptions of the concept of isomorphism of graphs.

In order to investigate how the students' challenge with the concept of graph isomorphism, a qualitative study was designed to analyze students' perceptions of isomorphism. This type of research can be useful to highlight similar structural features in many concepts in different mathematical fields. In this research, through the variation theory lens of Marton and Booth, the students' varying perceptions of graph isomorphism concept were explored in terms of attributes and conceptual metaphors.

In order to analyze students' perceptions of conceptions, first, variation theory is proposed as an explanation for how students learn. From this view, learning is treated as an individualized experience through which variations in concept instantiations provide the primary context for learning. (Marton and Booth 2013) Learning is defined as the perception of new attributes of a phenomenon or experience of a phenomenon in a new way. For a given individual, their understanding of a concept reflects which attributes of the concept are foregrounded for them. Through variation in instantiation, an individual may become aware of different properties. By comparing examples and non-examples, students may perceive attributes as either critical aspects or permissible variations. (Marton and Pang, 2006) a critical aspect is an attribute of a given concept that is invariant across all examples. In the case of graph isomorphism, students may not discern the critical aspect of two isomorphic graphs if they never encounter a non-example with a different isomorphic graph.

Permissible variations in contrast are features that can vary across examples of a given concept. If a student only experiences instantiations with a particular variation, they may overgeneralize that attribute to be a critical aspect. (Rupnow 2019) in the case of graphs isomorphism, a student may perceive a feature such as a connectivity as being critical if they never experience contrasting examples. Variation theory is a theory of learning and experience that explains how a learner might come to see, understand, or experience a given phenomenon in a certain way. In variation theory, it is assumed that there are Critical aspects of a given phenomenon that learners must simultaneously be aware of and focus on in order to experience that phenomenon in a particular way. Discernment of a critical aspect of a phenomenon result from experiencing variation in dimensions that correspond to that aspect.

Variation theory explains that individuals see, understand, and experience the world from their own perspectives (Orgill, 2012). Therefore, students may not learn effectively if they are not aware of things in exactly the same way as the teacher (Lo, 2012). The theory envisages that for learning to occur, some critical aspects of the object of learning must vary while other aspects remain constant (Ho, 2014; Ko & Marton, 2004; Marton & Booth, 1997). It further suggests that how students perceive a specific object of learning depends on what pattern of variation is provided by the teacher. It is expected that different patterns of variation result in different types of learning. Students' experiences with examples play a large role in their concept understanding. As such, even in the formal setting of graph theory, one would expect students to have perceived varied critical aspects of graph isomorphism beyond a given formal definition.

Vikstrom (2008) has expanded variation theory to incorporate metaphor as a way to make sense of this structuring. Metaphors provide a tool for concretizing abstract mathematical concepts. (Lakoff and Núñez, 2000)

Generally, metaphors are defined as a projection from a source domain into a target domain. For example, the familiar context of object collections (source) may serve as a metaphor for the mathematical context of arithmetic (target). Lokoff & Núñez distinguish between grounding metaphors, those metaphors that connect different mathematical domains. For the scope of this paper, Students' grounding metaphors for understanding graph isomorphism are the focus of this research. A students' perceptions of learning can be examined by attending to both their grounding metaphors and the critical aspects communicated with these metaphors, using Vikstrom's theory linking variation and metaphors.

A theoretical lens for analyzing mappings is the conceptual metaphor construct (Lakoff and Johnson 1980) Lakoff and Johnson (1980) posit that people's conceptual systems are metaphorical and that the metaphorical language individual use can be examined as evidence of the structure of their metaphorical system. Conceptual metaphors reveal the structure of thought, indicating they are a suitable lens for studying the abstract concepts of isomorphism. The function literature also provides insight into a number of metaphors that students could potentially leverage if attending to the function portion of the graph isomorphism definition. Lakoff and Nunez's (2000) categories include function is a machine and function is a collection of objects with directional links. Recently, Zandieh, Ellis, and Rasmussen (2017) categorized function metaphors used by linear algebra students including input/output, traveling, mapping, morphing and machine. The commonality across these categories is "an entity 1, an entity2, and a description about how these two are connected". When considering graph isomorphism, this can be likened to transitioning from entity1 (a vertex in G_1) to entity2 (a vertex in G_2). We speculate that some of these metaphors could be applicable in the context of graph isomorphism.

Representations provide another lens for parsing students' understanding of mathematical Concepts. If a student understands a concept, she should be able to flexibly engage different representations. Because so little has been explored related to graph isomorphism representations, literature on function provides insight into student treatment of representation. One of the most robust findings from this literature is that students often require or prefer explicit symbolic rules in event when other representation would be advantageous (knuth, 2000).

This research aims to contribute to the process of teaching mathematics through a complementary exploration of students perceived critical aspects (representational and definitional) and metaphors that organize and communicate their perceptions. Specifically, we will answer the following two questions:

1- What attributes do students treat as critical when engaging with graph isomorphism tasks?

2-what metaphors do students leverage when communicating about graph isomorphism and attributes?

Material and Methods

In line with the theoretical perspective, we designed a two-part study in which we confronted students with examples and non-examples of graph isomorphism through surveys and interviews. We started from the assumption that different students would have different experiences and perceive graph isomorphism differently in terms of metaphors and critical aspects.

The first part of our study consisted of a survey with different tasks aimed at students' perceptions of learning. The survey was conducted at two universities were graph theory is studied at introductory level ($n = 12$, $n = 12$).

Both classes focused on graph theory and provided students with a definition of graph isomorphism. The survey was designed based on the literature search and teaching experiences of graph theory (e.g, Hammack, Gross and Yellen and Anderson, 2019). Students had opportunities to engage with graph isomorphism examples and non-examples through the following activities:

1. Determining if a given instantiation is an example of a concept.
2. Determining if two examples are mathematically the same.
3. Determining what properties and example may or may not have.
4. Generating an example meeting some criteria.

We speculated that Activity 1 would reveal critical aspects related to students' personal definition. Activity 2 would uncover critical aspects that determined how students perceived the nature of two isomorphic graphs. Activity 3 would uncover critical aspects related to notation and Activity 4 would uncover implicit aspects treated as critical that can constrain example generating. Every

[DOI: 10.22034/4.1.155]

activity was open-ended and provided a prompt for the students to explain their reasoning. The surveys contained 15 Common questions. (table1) In addition to the surveys, we conducted semi-structured interviews with each of 6 participants to gain deeper insight into the students' mathematical thinking they worked through the survey. Our analyses were conducted in two phases using phenomenographic (Trigwell, 2006) analysis. First, we wanted to identify a set of perceived critical aspects that varied across the 24 survey responders. Through the literature review and our prior studies, we identified the following attributes: bijective, adjacency-preserving and non-adjacency preserving.

We created a series of tasks to determine if students considered these attributes to be critical or variable. However, we were also open to the possibility of other attributes emerging from the responses to the tasks. After analyzing the surveys related to the initial set of attributes, we discovered additional attributes that were viewed as critical by some students and variable by others.

Table 1. Survey tasks

Activity	Examples
Determine if a graph isomorphism	Task 1a:

Graph	Vertex Mappings							
G	a	B	c	d	e	f	g	h
H	i	J	n	m	l	k	o	p

Graph	Edge Mappings											
G	ab	ad	ae	bf	bc	fg	ef	eh	gh	cd	cg	dh
H	ij	im	il	jk	jn	ko	lk	lp	op	nm	no	mp

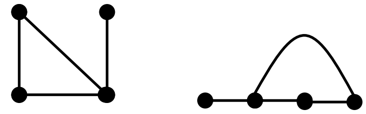
Task 1b: A function $f: G \to H$ from a graph G to a graph H is a map $f: V(G) \to V(H)$ for which $xy \in E(G)$ inplies $f(x)f(y) \in E(H)$.

Task 1c: The vertex function $j \to j + 4$ is bijective and adjacency-preserving, but dose not preserve non-adjacency.

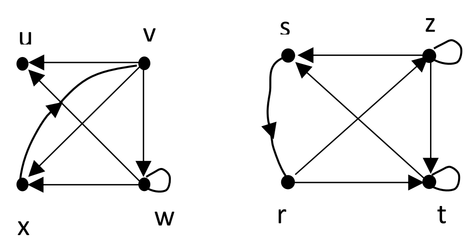
Task 1d: The vertex function $j \to j$ mod 2 preserves adjacency and non-adjacency but it is not bijective.

same Or Different

Task 2a:



Task 2b:



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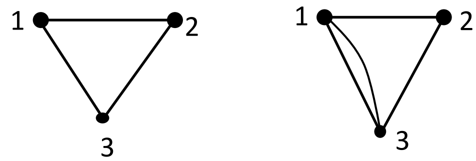
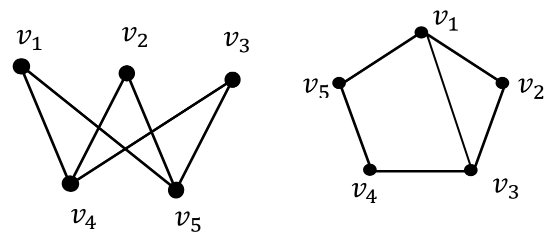
Task 2c:

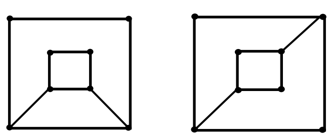


Task 2d:

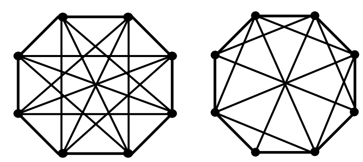


Property Check — Task 3: which property do two graphs H and G have in common?

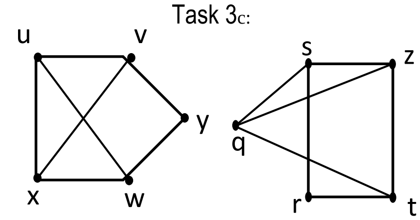
Task 3a:



Task 3b:



Task 3c:



Generate Example

Task 4a: Give an example of two non-isomorphic 9-vertex graphs with the same degree sequence.

Task 4b: Find all possible isomorphism types of the tree with 5 vertices.

Task 4c: Draw all isomorphism types of simple graph with four vertices and three edges.

Task 4d: How many isomorphism are there from $k_{m,m}$ to itself?

To situate students perceived critical aspects and permissible variations, we conducted a more thorough analysis of the six interviews using Larsson and Holmstrom's (2007) phenomenographic process which relies on a primary reader and co-reader to develop metaphor profiles.

Both readers read through the entirety of the transcribed interviews, identified instances where graph isomorphism was addressed, and then analyzed these excerpts for perceived critical aspects and metaphors. This analysis led to the identification of three metaphor categories: Sameness, Sameness / mapping and Formal definition. For each interview participant, we created profiles including descriptions of their metaphor categories and perceived critical aspects and permissible variations for graph isomorphism (table 2)

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Table 2. Profile of students participating in the interview

	Sara	Behzad	Arman	Elham	Arya	Elya
Critical aspect						
Adjacency-preserving	Y	Y	Y	Y	Y	N
Non- adjacency-preserving	N	N	Y	Y	N	Y
Bijective	N	N	N	Y	Y	N
Permissible variation						
Degree sequence	Y	N	Y	Y	N	N
Circuit and path	N	N	Y	Y	N	Y
Incidence matrices	N	N	N	N	N	N
Metaphor category[1]						
Sameness	Dominant	Dominant	Dominant	Dominant		Dominant
Sameness/mapping	Non-dominant	Non-dominant	Dominant	Non-dominant	Non-dominant	
Formal definition	Non- Dominant		Dominant		Dominant	

Y: Treated critical aspect as critical or permissible variation as permissible;

N: Inappropriate treatment of attribute;

[1] Empty cells indicate that metaphor was not evoke by the student during the interview;

Results

Through our analysis, we found that students exhibited different treatment of six key attributes: adjacency-preserving, non-adjacency preserving and bijective. Table 1 contains the descriptions of each attribute, and the frequency of students treating the attribute as critical (a consistent attribute of isomorphism) or variable (an attribute that an isomorphism may or may not have). For each of these attributes, at least a third of student responses diverged in their treatment of features

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as critical or variable. In the next subsections, we share interview data illustrating variable and critical or variable.

Additionally, during the follow-up interviews, we identified three categories of metaphors students communicated related to graph isomorphism and its attributes: Sameness, Sameness/mapping and Formal definition. Table3 provides an overview of the metaphor types and attribute profiles of the interview participants.

Table 3 . Frequency of critical aspects and permissible variation from the survey

Attribute	Description	Indicator in response	Frequency ($n = 24$)
Critical aspect: $n = 18$ Adjacency – preserving	A vertex function $f: V_G \rightarrow V_H$ preserves adjacency if for every pair of adjacent vertices $u\ and\ v$ in graph G, the vertices $f(u)\ and\ f(v)$ are adjacent in graph H .	Task 1	Treated as critical Treated as variable $n = 6$
Critical aspect: $n = 15$ Non-adjacency preserving	f preserves non-adjacency if $f(u)\ and\ f(v)$ are non-adjacency whenever $u\ and\ v$ are non-adjacent.	Task 1	Treated as critical Treated as variable $n = 9$
Critical aspect: $n = 9$ Bijective $n = 15$	A bijective function $f: X \rightarrow Y$ is a one- to- one (injective) and onto (surjective) mapping of a set X to a set Y .	Task 4	Treated as critical Treated as variable
Permissible variation: $n = 7$ Degree sequence $n = 17$	A non-increasing sequence of the vertex degree of the graph vertices, so with repetitions as needed.	Task 3	Treated as critical Treated as variable
Permissible variation: $n = 5$ Circuit and path $n = 19$	A circuit is a sequence of adjacent nodes starting and ending at the same node. A path is a sequence of non-repeated nodes connected through edges present in a graph.	Task 2	Treated as critical Treated as variable

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Permissible variation $n = 2$	the incidence matrix of a directed graph is a $n \times m$	Task 2	Treated as critical
Incidence matrix $n = 22$	matrix B where n and m are the number of vertices		Treated as variable
	and edges respectively, such that $B_{ij} = \begin{cases} -1 & if\ adge\ e_j\ leaves\ vertices\ v_i\ , \\ 1 & if\ adge\ e_j\ inters\ vertices\ v_i\ \ , \\ 0 & otherwise. \end{cases}$		

Sameness metaphors

The first category we unpack is that of sameness metaphors.

Metaphor category	metaphor code	metaphor definition
Sameness	Generic sameness	Generic references to graphs being the same or similar, whether at the whole graph level or as general statements about relationships between vertices.
	Same properties	Use of properties that are same for all isomorphic graphs. (e.g., cardinality , same degree sequence , Same edge connectivity).
	Disembedding	Structure – focused language to high light (Sub) structures for special inspection by the existence of an isomorphism.

Many examples of sameness language were invoked in both interview and task-based surveys, though more variety appeared in interview. In the interview setting, students often invoked generic sameness.

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Their initial description of isomorphism was: "when I think about graphs, if they are isomorphic, it means that they are the same graph just notated with different names or notated with a different operation, but that the graphs are essentially the same." " … The heart of the matter is that they [isomorphic graphs] are actually the same." And "if I'm saying 2 graphs are the same, they should have the same number of vertexes. That's a pretty low criterion for being the same." Disembedding skills are the spatial skills needed to separate one object or picture from a more complicated background. This allows us to understand how complicated structures are made up from separate parts.

Sameness/ mapping metaphors

The second category we will discuss is that of sameness/ mapping metaphors.

Metaphor category	metaphor code	metaphor definition
Sameness/mapping	Renaming/Relabeling	Giving new names or labels to vertices to show equivalence between graphs
	Matching	Connecting specific vertices in two graphs or lining up vertices in order to create a specific correspondence that reveals sameness of the paired vertices
	Generic mapping	Generic reference to an isomorphism as a function or mapping without further details about the mapping or explicit reliance on properties of functions.
	Function/journey/machine	Specific use of a function property, traveling from a starting point to an ending point, connections to how a machine works(e.g. takes inputs and produces outputs).

Mapping category language was ubiquitous, but was only central to discussions about the nature of isomorphism being functions. Sameness / mapping category language was used when initially defining and throughout tasks but was used less after the formal definition was given. Generic mapping was used when looking for formulaic representations of isomorphism or when seeing what to "map to" specific elements. Journey metaphors that referenced going from one graph to another or elements being "sent" were used in multiple periods. The greater variety of mapping category metaphors in interviews than in task-based surveys is likely because of the difference in the types of questions posed and instructional goals. The difference between metaphors in the interviews and in task-based surveys can be explained by overarching conceptual questions being asked in the interview versus the variety of activities in task-based surveys.(e.g. provide examples, prove intuition, define relevant terms).

Formal definition metaphors

The third type of metaphor category to emerge was formal definition metaphors.

Metaphor category	Metaphor code	Metaphor definition
Formal definition	Literal formal definition	Use of the string of symbols in the formal definition for isomorphism or use of words related to bijective, onto , or one-to-one (or a mapping lacking those properties) to talk about graph isomorphism.
	Structure-preserving	Use of "structure-preserving" or a slight variation without interpretation.
	Operation-preserving	Use of "operation-preserving" or a slight variation without interpretation or use of a specific operation while talking about Preserving (preserving addition).

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The literal formal definition was used when discussing why isomorphism should require being one and onto but was mostly used with proofs.

Operation-preserving language was used a few times to summarize what happened in the isomorphism property structure–preserving language was introduced by a student in interview to note a “same structure” being shared, at which point the student noted “structural differences” would indicate graphs were not isomorphism. Otherwise, this metaphor was not observed. The formal definition category was present in both the interview and task-based surveys. However, it was not a focal point in either context.

While time was spent in task-based surveys developing the informal ideas around sameness into the formal definition, the way students were encouraged to think about isomorphism was still rooted in sameness.in the interview, the definition was also mentioned in passing. But more time was spent thinking about what that meant, largely in terms of sameness.

Discussion

According to variation theory, identifying and distinguishing between critical aspects of a concept and then combining them together, provide the best strategies for understanding a concept. The purpose of this study was to investigate the students' perceptions of the concept of graphs isomorphism and to identify the conceptual metaphors used by them. First the critical and variable aspects of the concept of graph isomorphism were identified. Then, students’ attention to these critical aspects in task-based survey and interviews were analyzed. The results showed that the students do not understand the critical aspect of bijective mapping in defining the graph isomorphism, such neglect of this feature may make it difficult to prove theorems and understand the concept. Nardi (2000) noted students’ struggles in proving the isomorphism theorems stemmed from three major sources: an inability to recall definitions or a lack of understanding of definitions, poor conceptions of mapping, and not recognizing the purpose of sections of the proof. In this study, we clearly showed that students are not familiar with a number of representations related to the concept of graph isomorphism. Furthermore, students could not establish a relationship between different representations of graph isomorphism and had a weak conceptual understanding of graph isomorphism. This topic can be considered in teaching of isomorphism concept. The second contribution was an exploratory analysis of the metaphors that may organize and communicate these attributes. We identified sameness, sameness/mapping and formal definition metaphors from this analysis. The sameness notion highlighted by Leron et al. (1995) and noted by other researchers (e.g., Weber & Alcock, 2004) occurred frequently here as well (Generic Sameness). Other, more specific language like relabeling has been documented as relevant to isomorphism for mathematicians (Weber & Alcock, 2004) and is apparent here in the Renaming/Relabeling metaphor. Revisiting research questions, students intentionally drew upon ideas of sameness to discuss isomorphism in interview and task-based survey. These included calling isomorphic essentially the same and using Renaming/Relabeling to talk about how a function represents the isomorphism. Only two students used the formal definition metaphor in a secondary capacity. Both made use of mapping metaphors while discussing mapping but did not seem to view this language as the main conceptual point of isomorphism. This seems to be because they felt the structures themselves (being isomorphic) were more important than the mappings connecting them (isomorphism).

[DOI: 10.22034/4.1.155]

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Ethics statement

The studies involving human participants were reviewed and approved by the ethics committee of Islamic Azad University. The patients/participants provided their written informed consent to participate in this study.

Author contributions

All authors contributed to the study conception and design, material preparation, data collection, and analysis. All authors contributed to the article and approved the submitted version.

Acknowledgments

We would like to express our gratitude to the instructors and students Ardebil University for their participation in this study.

Funding

The authors did (not) receive support from any organization for the submitted work.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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